# Contents

# Chapter 13

# Vector Valued Functions and Motions in Space

In this chapter we study two types of special functions:

(1) Continuous mapping of one variable(called a **curve**)

(2) Mapping from a subset of $\mathbb{R}^n$ to itself(called **vector fields**)

## 13.1 Curves and Tangents

When a particles moves in the space during a times interval $I$, we think of its coordinates as a vector function $\mathbf{r}(t) = (f(t), g(t), h(t))$ defined on $I$. The points $(x, y, z) = (f(t), g(t), h(t))$ make up a curve called a **path**.

**Definition 13.1.1.** A **curve**(or **path**) can be represented as a function $\mathbf{r} : I = [a, b] \to \mathbb{R}^n, n = 2, 3$. It is called a **parameterized curve**. $\mathbf{r}(a)$ and $\mathbf{r}(b)$ are called the **endpoints** of the path.

A parameterized curve $\mathbf{r}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ can be written as

$$\mathbf{r}(t) = \vec{OP} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} = (x_1(t), x_2(t), x_3(t)). \tag{13.1}$$

$f(t), g(t), h(t)$ are called **component functions**. It may be viewed as the position of a particle moving along the curve.

A function having vector value, like equation (13.1) is called a **vector valued function**.

We define the limit of a vector function as

$$\lim_{t\to t_0} \mathbf{r}(t) = \mathbf{L} = (\lim_{t\to t_0} f(t), \lim_{t\to t_0} g(t), \lim_{t\to t_0} h(t)).$$

**Definition 13.1.2.** If all the component $x_i(t)$ of $\mathbf{r}$ is continuous(resp. differentiable), then we say $\mathbf{r}$ is **continuous(resp. differentiable)** and its derivative is written as

$$\mathbf{r}'(t) = \lim_{\Delta t\to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} = (f'(t), g'(t), h'(t)). \tag{13.2}$$

## The geometric meaning of derivative of $\mathbf{r}(t)$

When $\mathbf{r}(t) \neq 0$, it represents a **tangent vector** at $t$.

**Definition 13.1.3.** A curve $\mathbf{r}(t)$ is said to be **smooth** if $d\mathbf{r}/dt$ is continuous and never zero. In this case, the image curve looks smooth. One of the reason for requiring nonzero derivative is that we want to avoid the case when a particle moving along the curve traces back. (i.e., move backward)

On a smooth curve, there is no sharp corner or cusps.

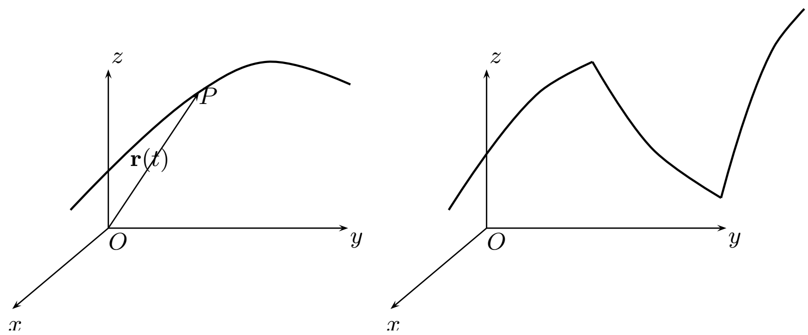
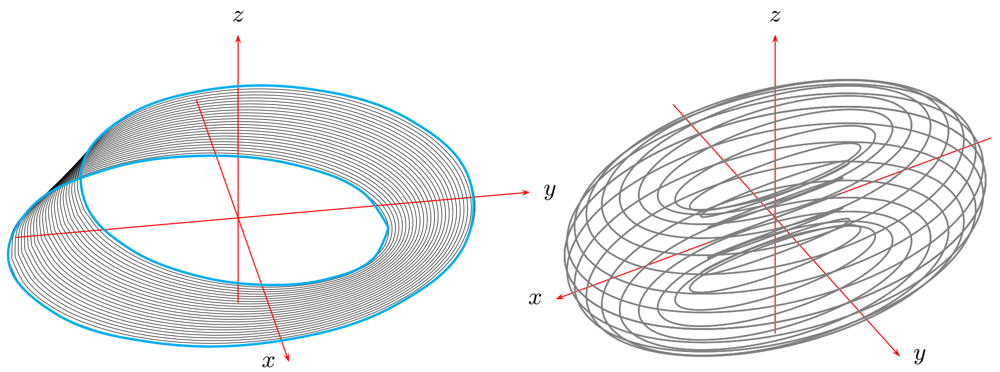



Figure 13.1: piecewise smooth curve can have no tangent at cusps

A path may have many **parametrizations**.

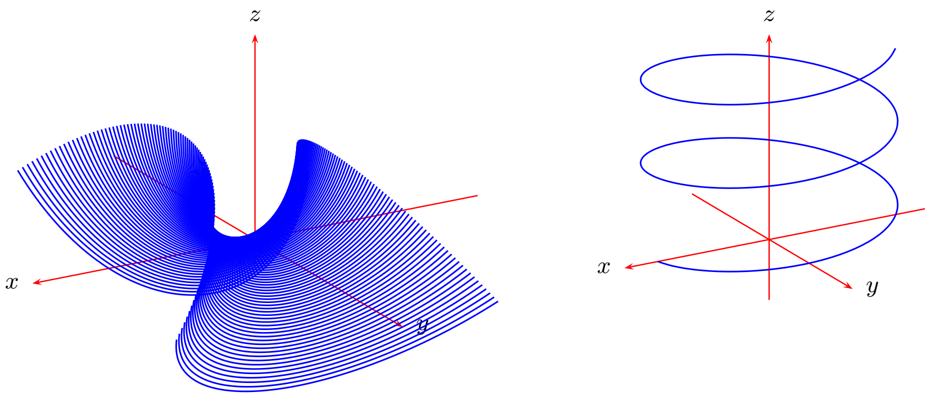
**Example 13.1.4.** (1) $\mathbf{r}(t) = \mathbf{a} + t\mathbf{b}$ is a line

(2) $\mathbf{r}(t) = (\cos t, \sin t)$ on $[0, 2\pi]$ is path traveling a circle once. If the domain is $[0, 4\pi]$, it travels twice.




$\left(\left(1+\frac{1}{2}\cos\frac{t}{2}\right)\cos t,\ \left(1+\frac{1}{2}\cos\frac{t}{2}\right)\sin t,\ \frac{1}{2}\sin\frac{t}{2}\right)$ $((a+b\cos 20t)\cos u, (a+b\cos 20t)\sin u, b\sin 20t)$

Figure 13.2: Graph of Möbius strip and torus




Enneper's Surface $(u-\frac{u^3}{3}+uv^2, v-\frac{v^3}{3}+vu^2, u^2-v^2)$ Helix : $(\cos t, \sin t, t)$.

Figure 13.3: Family of curves

(3) A family of curves are obtained from surface: If we fix say $v = 1$ from Enneper's surface, we get $(2u - \frac{u^3}{3}, \frac{2}{3} + u^2, u^2 - 1)$. (Fig 13.3)

(4) $\mathbf{r}(t) = (a\cos t, a\sin t, bt)$ defines a circular helix. (Fig 13.3)

## Derivatives and Motion

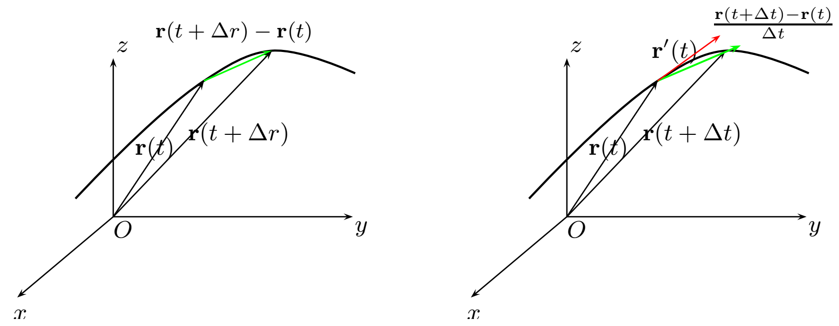



Figure 13.4: As $\Delta t \to 0$, $\mathbf{r}'(t)$ becomes tangent vector

**Definition 13.1.5.** Let $\mathbf{r}$ be a smooth curve. Then

(1) the **velocity** is defined : $\mathbf{v}(t) = \mathbf{r}'(t)$

(2) the **speed** of $\mathbf{r}$ is $\|\mathbf{v}(t)\|$.

(3) the **acceleration** vector is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

(4) the unit vector $\mathbf{v}(t)/\|\mathbf{v}(t)\|$ is the direction.

**Proposition 13.1.6.** *Let* $\mathbf{r}$ *be a differentiable path and assume* $\mathbf{v}_0 = \mathbf{v}(t_0) \neq 0$. *The tangent line to the path is given by*

$$\ell(t) = \mathbf{r}_0 + (t - t_0)\mathbf{v}_0. \tag{13.3}$$

**Example 13.1.7.** Find the velocity, acceleration of a particle moving along the curve $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + 4\cos^2 t\mathbf{k}$.

**sol.**

$$\mathbf{v}(t) = \mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} - 8\cos t\sin t\mathbf{k} = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} - 4\sin 2t\mathbf{k}.$$

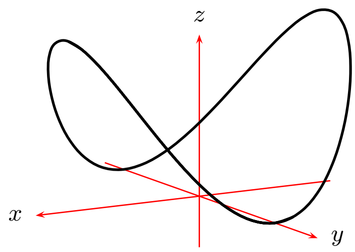



Figure 13.5: Curve of Example 13.1.7

The acceleration is

$$\mathbf{a}(t) = \mathbf{r}''(t) = -2\cos t\mathbf{i} - 2\sin t\mathbf{j} - 8\cos 2t\mathbf{k}$$

The speed is

$$\|\mathbf{v}(t)\| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (-4\sin 2t)^2} = \sqrt{4 + 16\sin^2 t}.$$

The position when $t = 7\pi/4$ is

$$\mathbf{r}(\frac{7\pi}{4}) = 2\cos\frac{7\pi}{4}\mathbf{i} + 2\sin\frac{7\pi}{4}\mathbf{j} + 4\cos^2\frac{7\pi}{4}\mathbf{k} = \sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j} + 2\mathbf{k}.$$

The velocity vector at $t = 7\pi/4$ is

$$\mathbf{v}(\frac{7\pi}{4}) = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} + 4\mathbf{k}.$$

□

**Example 13.1.8.** A particle moves with a constant acceleration $\mathbf{a}(t) = -\mathbf{k}$. When $t = 0$ is the position is $(0, 0, 1)$ and velocity is $\mathbf{i} + \mathbf{j}$. Describe the motion of the particle.

**sol.** Let $\mathbf{c}(t) = (x(t), y(t), z(t))$ represent the path traveled by the particle. Since the acceleration is $\mathbf{c}''(t) = -\mathbf{k}$ we see the velocity is

$$\mathbf{c}'(t) = C_1\mathbf{i} + C_2\mathbf{j} - t\mathbf{k} + C_3\mathbf{k}.$$

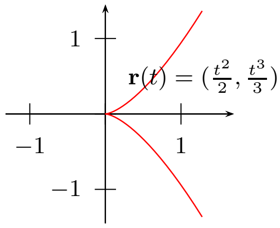



Figure 13.6: At a cusp, $\frac{d\mathbf{r}(t)}{dt}|_{t=0} = 0$

Hence by initial condition, $\mathbf{c}'(t) = \mathbf{i} + \mathbf{j} - t\mathbf{k}$ and so $\mathbf{c}(t) = t\mathbf{i} + t\mathbf{j} - \frac{t^2}{2}\mathbf{k} + Const\ vec$. The constant vector is $\mathbf{k}$. Hence $\mathbf{c}(t) = t\mathbf{i} + t\mathbf{j} + (1 - \frac{t^2}{2})\mathbf{k}$.

□

**Example 13.1.9.** The image of $C^1$-curve is not necessarily "smooth". it may have sharp edges; (Fig 13.6)

(1) Cycloid: $\mathbf{c}(t) = (t - \sin t, 1 - \cos t)$ has cusps when it touches $x$-axis. That is, when $\cos t = 1$ or when $t = 2\pi n, n = 1, 2, 3, \cdots$.

(2) Hypocycloid: (Fig 13.7) $\mathbf{c}(t) = (\cos^3 t, \sin^3 t)$ has cusps at four points when $\cos t = 0, \pm 1$.

(3) Consider $\mathbf{r}(t) = (\frac{t^2}{2}, \frac{t^3}{3})$. Eliminating $t$, we get

$$(2x)^3 = (3y)^2.$$

At all these points, we can check that $\mathbf{c}'(t) = 0$.(Roughly speaking, tangent vector has no direction or does not exist.)

**Differentiation Rules**

(1) $\frac{d}{dt}[\mathbf{b}(t) \pm \mathbf{c}(t)] = \mathbf{b}'(t) \pm \mathbf{c}'(t)$ (Sum/difference)

(2) $\frac{d}{dt}[p(t)\mathbf{c}(t)] = p'(t)\mathbf{c}(t) + p(t)\mathbf{c}'(t)$ for any differentiable scalar function $p(t)$ (scalar multiple)

(3) $\frac{d}{dt}[\mathbf{b}(t) \cdot \mathbf{c}(t)] = \mathbf{b}'(t) \cdot \mathbf{c}(t) + \mathbf{b}(t) \cdot \mathbf{c}'(t)$ (dot prodcut)

(4) $\frac{d}{dt}[\mathbf{b}(t) \times \mathbf{c}(t)] = \mathbf{b}'(t) \times \mathbf{c}(t) + \mathbf{b}(t) \times \mathbf{c}'(t)$ (cross product)

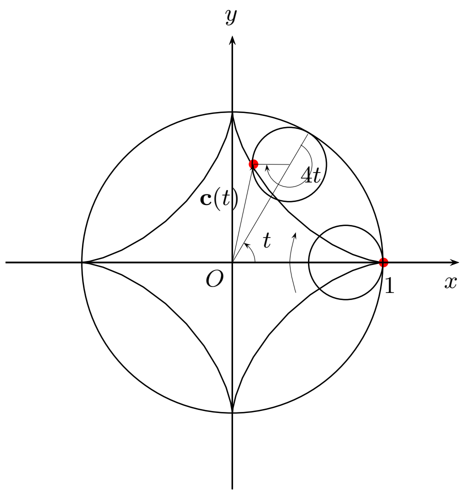



Figure 13.7: Hypocycloid $x^{2/3} + y^{2/3} = 1$

(5) $\frac{d}{dt}[\mathbf{c}(q(t))] = q'(t)\mathbf{c}'(q(t))$ (chain rule)

**Example 13.1.10.** Figure 13.7. A circle $C$ of radius 1/4 is rolling along the unit circle $U\colon x^2 + y^2 = 1$. Represent the locus of the point $P$ starting from $(1, 0)$ to return to itself.

**sol.** (Refer to Figure 13.7). The center of $C$ is $\frac{3}{4}(\cos t, \sin t)$, The desired point $\mathbf{c}(t)$ is given by

$$\begin{aligned}\mathbf{c}(t) &= \frac{3}{4}(\cos t, \sin t) + \frac{1}{4}(\cos(t-4t), \sin(t-4t)) \\ &= \frac{1}{4}(3\cos t + \cos 3t, 3\sin t - \sin 3t)\end{aligned}$$

Since $\mathbf{c}(2\pi) = (1, 0)$, we see the path is

$$\mathbf{c}(t) = \frac{1}{4}(3\cos t + \cos 3t, 3\sin t - \sin 3t), \quad 0 \le t \le 2\pi$$

by trig. identity [1] it beceoms

$$\mathbf{c}(t) = (\cos^3 t, \sin^3 t), \quad 0 \le t \le 2\pi$$

or

$$x^{2/3} + y^{2/3} = 1$$

[1] $\sin 3\theta = 3\sin\theta - 4\sin^3\theta, \quad \cos 3\theta = 4\cos^3\theta - 3\cos\theta$

□

## 13.2 Integrals of Vector functions; Projectile Motion

If the component of $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is integrable over $[a, b]$ then we can define its integral as follows

$$\boxed{\int_a^b \mathbf{r}(t)dt = \left(\int_a^b f(t)dt\right)\mathbf{i} + \left(\int_a^b g(t)dt\right)\mathbf{j} + \left(\int_a^b h(t)dt\right)\mathbf{k}}$$

### 13.2.1 Projectile Motion

**Example 13.2.1** (Throwing a ball)**.** Assume a baseball a player throws a ball(or a cannon ball) with an initial velocity $\mathbf{v}_0$m/sec that is in the direction of $(\cos\alpha, \sin\alpha)$. Describe the trajectory.

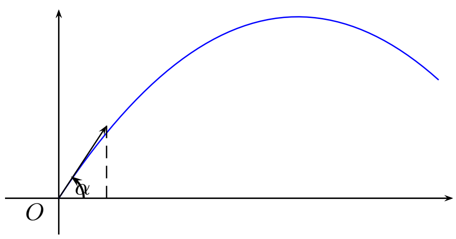



Figure 13.8: A projectile

sol. The motion follows from Newton's second law of motion:

The force acting on the ball is equal to the mass times the acceleration: $\mathbf{F} = m\mathbf{a}$.

Since the acceleration is $\mathbf{a}(t) = \mathbf{r}''(t)$, we must have

$$m\mathbf{a} = m\mathbf{r}''(t) = -mg\mathbf{j} \text{ or } \mathbf{r}''(t) = -g\mathbf{j},$$

where $g = 9.8m^2/sec$ is the gravity constant.

Integrating, we get the velocity

$$\mathbf{v}(t) := \mathbf{r}'(t) = -gt\mathbf{j} + \mathbf{c}$$

for some constant vector $\mathbf{c}$. Integrating once more, we obtain

$$\mathbf{r}(t) = -\frac{1}{2}gt^2\mathbf{j} + \mathbf{c}t + \mathbf{d}.$$

Since the initial velocity is $\mathbf{v}(0) = \mathbf{c} = 20(\cos\alpha, \sin\alpha)$, $v_0 = 20$ we have

$$\mathbf{r}(t) = -\frac{1}{2}gt^2\mathbf{j} + (v_0\cos\alpha)t\mathbf{i} + (v_0\sin\alpha)t\mathbf{j} + \mathbf{d},$$

where $\mathbf{d}$ is the initial position of the ball.

□

**Example 13.2.2** (baseball hit)**.** A baseball is hit when it is 1 m above the ground. The initial speed if 50m/s at an angle of 20 degrees (with horizontal). At the moment of hit, the wind was blowing in the opposite direction of the ball $2.5m\mathbf{i}/s$.

(1) Find the location

(2) How high does the ball go and when it reaches its maximum height?

(3) How far it would go until it hits the ground and when ?

**sol.** The situation is the same as above example except the effect of wind. So

$$\begin{aligned}\mathbf{r}(t) &= -\frac{1}{2}gt^2\mathbf{j} + (v_0\cos\alpha - 2.5)t\mathbf{i} + (v_0\sin\alpha)t\mathbf{j} + \mathbf{j}\\ &= (50\cos 20^o - 2.5)t\mathbf{i} + (1 + 50\sin 20^o t - 4.9t^2)\mathbf{j}.\end{aligned}$$

it reaches maximum when $dy/dt = 50\sin 20^o - 9.8t = 0$, $t = 1.75$.

□

**Example 13.2.3.** Show that if $\mathbf{c}(t)$ is a vector function such that $\|\mathbf{c}(t)\|$ is constant, then $\mathbf{c}'(t)$ is perpendicular to $\mathbf{c}(t)$ for all $t$.

Solution:

$\|\mathbf{c}(t)\|^2 = \mathbf{c}(t) \cdot \mathbf{c}(t)$. Derivative of constant is zero. Hence

$$0 = \frac{d}{dt}[\mathbf{c}(t) \cdot \mathbf{c}(t)] = \mathbf{c}'(t) \cdot \mathbf{c}(t) + \mathbf{c}(t) \cdot \mathbf{c}'(t) = 2\mathbf{c}(t) \cdot \mathbf{c}'(t).$$

Thus $\mathbf{c}'(t)$ is perpendicular to $\mathbf{c}(t)$.

□

## 13.3 Arc Length

**Definition 13.3.1** (Arc Length)**.** Suppose a curve $C$ has one-to-one differentiable parametrization $\mathbf{r}$. Then the **arc length** is defined by

$$L(\mathbf{r}) = \int_a^b \|\mathbf{v}(t)\|\, dt = \int_a^b \|\mathbf{r}'(t)\|\, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}dt.$$

To find the length of a path, we divide the path into small pieces and approximate each piece by a line segment joining the end points; then summing the length of individual line segments we obtain an approximate length. The length is obtained by taking the limit. To define it precisely, we use the Riemann integral.

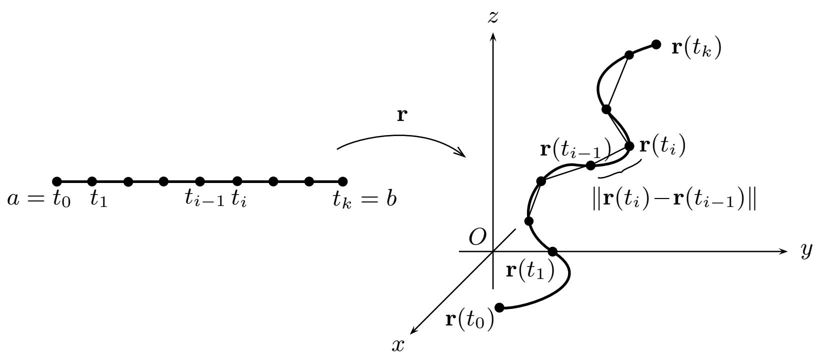



Figure 13.9: Riemann sum of the curve length

The sum of the line segment is

$$\begin{aligned}\sum_{i=1}^{k} \Delta s_i &= \sum_{i=1}^{k} \|\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})\| \\ &= \sum_{i=1}^{k} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2} \\ &= \sum_{i=1}^{k} \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta z_i}{\Delta t_i}\right)^2} \Delta t_i.\end{aligned}$$

As $k \to \infty$ it converges to

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt. \tag{13.4}$$

**Example 13.3.2.** Find the length of the curve $x^{2/3} + y^{2/3} = 1$.

**sol.** It suffices to consider the first quadrant and we parameterize it as

$$\mathbf{r}(t) = (\cos^3 t, \sin^3 t), \quad 0 \le t \le \pi/2.$$

$$\|\mathbf{r}'(t)\| = \sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2} = 3|\sin t \cos t|$$

Length is

$$\begin{aligned}4\int_0^{\pi/2} 3|\sin t \cos t|\, dt &= 6\int_0^{\pi/2} \sin 2t\, dt \\ &= 6\left[-\frac{1}{2}\cos 2t\right]_0^{\pi/2} \\ &= 6\left[-\frac{1}{2}(-1) - \left(-\frac{1}{2}\right)\right] \\ &= 6\end{aligned}$$

□

**Example 13.3.3.** Find the arclength of the helix $\mathbf{r}(t) = (a\cos t, a\sin t, bt)$, $0 \le t \le 2\pi$.

Sol.

$$\|\mathbf{r}'(t)\| = \| - a\sin t\mathbf{i} + a\cos t\mathbf{j} + b\mathbf{k}\| = \sqrt{a^2 + b^2}.$$

Hence

$$L(\mathbf{r}) = \int_0^{2\pi} \sqrt{a^2+b^2}dt = 2\pi\sqrt{a^2+b^2}.$$

**Example 13.3.4.** Find the arclength of the curve $(\cos t, \sin t, t^2)$, $0 \le t \le 2\pi$.

Sol.

$$\|\mathbf{v}\| = \sqrt{1+4t^2} = 2\sqrt{t^2+\frac{1}{4}}.$$

To evaluate this integral we need a table of integrals:

$$\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}[x\sqrt{x^2+a^2} + a^2\log(x+\sqrt{x^2+a^2})] + C.$$

**Example 13.3.5.** Find the length of the cycloid

$$\mathbf{r}(t) = (t - \sin t, 1 - \cos t).$$

Since

$$\|\mathbf{r}'(t)\| = \sqrt{(1-\cos t)^2 + (\sin t)^2} = \sqrt{2-2\cos t}$$

we see

$$\begin{aligned} L(\mathbf{r}) &= \int_0^{2\pi} \sqrt{2-2\cos t}dt = 2\int_0^{2\pi}\sqrt{\frac{1-\cos t}{2}}\,dt \\ &= 2\int_0^{2\pi} \sin\frac{t}{2}\,dt \\ &= 4\left(-\cos\frac{t}{2}\right)\Big|_0^{2\pi} = 8. \end{aligned}$$

**Example 13.3.6.** Suppose a function $y = f(x)$ given. Then the graph is viewed as a curve parameterized by $t = x$ and $\mathbf{r}(x) = (x, f(x))$. So the length of the graph from $a$ to $b$ is

$$L(\mathbf{r}) = \int_a^b \sqrt{1+(f'(x))^2}dx.$$

## Velocity and speed

Assume the path $\mathbf{r}(t) = (x(t), y(t), z(t))$ represents the movement of an object. In other word, the location of the object at time $t$ is given by $\mathbf{r}(t)$. Then the **instantaneous velocity** at $t = t_0$ is given as follows, and it is the tangent vector at $t = t_0$.

$$\mathbf{r}'(t_0) = \lim_{h\to 0} \frac{\mathbf{r}(t_0+h) - \mathbf{r}(t_0)}{h} = (x'(t_0), y'(t_0), z'(t_0)).$$

**Example 13.3.7.** If an object follow moving along the curve $\mathbf{c}(t) = t\mathbf{i} + t^2\mathbf{j} + e^t\mathbf{k}$ at time $t$ takes off the curve at $t = 2$ and travels for 5 seconds. Find the location.

**sol.** We assume the object travels along the tangent line after taking off the curve. The velocity at $t = 2$ is $\mathbf{c}'(2) = \mathbf{i} + 4\mathbf{j} + e^2\mathbf{k}$. Hence the location 5 second after taking off the curve

$$\begin{aligned}\mathbf{c}(2) + 5\mathbf{c}'(2) &= 2\mathbf{i} + 4\mathbf{j} + e^2\mathbf{k} + 5(\mathbf{i} + 4\mathbf{j} + e^2\mathbf{k}) \\ &= 7\mathbf{i} + 24\mathbf{j} + 6e^2\mathbf{k}.\end{aligned}$$

Hence the location is $(7, 24, 6e^2)$.

□

### Arc-Length Parameter

Recall : Given a $C^1$-parametrization of a curve $C$. Then we have seen that the **arc length** of $C$ is given by

$$L(\mathbf{r}) = \int_a^b \|\mathbf{r}'(t)\|\, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt.$$

**Definition 13.3.8.** Now we fix a base point $P = P(t_0)$ and treat upper limit of the integral as a variable $t$. Then the arclength becomes a function of $t$, **arc-length function** :

$$s(t) = \int_{t_0}^t \|\mathbf{r}'(\tau)\|\, d\tau.$$

The arc-length (parameter)function satisfies

$$\frac{ds}{dt} = s'(t) = \|\mathbf{r}'(t)\| = speed.$$

Assuming $\mathbf{r}'(t) \neq 0$, we see $\frac{ds}{dt}$ is always positive. Hence we can solve for $s$ in terms of $t$(inverse function theorem). Hence we can use $s$ as a new parameter to represent the curve $C$.

**Example 13.3.9.** For the helix $\mathbf{r}(t) = (a\cos t, a\sin t, bt)$, we can find a new parametrization by $s$ as follows:

$$s(t) = \int_0^t \|\mathbf{r}'(\tau)\|\, d\tau = \int_0^t \sqrt{a^2+b^2} d\tau = \sqrt{a^2+b^2}\ t,$$

so that

$$s = \sqrt{a^2+b^2}\ t, \text{ or } t = \frac{s}{\sqrt{a^2+b^2}}.$$

Hence

$$\mathbf{r}(t(s)) = \left(a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{bs}{\sqrt{a^2+b^2}}\right).$$

**Definition 13.3.10.** The **unit tangent vector T** of the path $\mathbf{r}$ is the normalized velocity vector

$$\mathbf{T} = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}.$$

**Example 13.3.11.** For the helix $\mathbf{r} = (a\cos t, a\sin t, bt)$, we have

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{-a\sin t\mathbf{i} + a\cos t\mathbf{j} + b\mathbf{k}}{\sqrt{a^2+b^2}}.$$

**Example 13.3.12.** For the curve $\mathbf{r} = (t, t^2, t^3)$, we have

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}.$$

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}}{\sqrt{1+4t^2+9t^4}}.$$

But arclength is not easy to compute:

$$s(t) = \int_0^t \sqrt{1+4t^2+9t^4}\, dt.$$

**Example 13.3.13** (Change of the position $\mathbf{r}$ vector w.r.t arclength)**.** In general, finding a parametrization by arclength parameter $s$ is not a simple task. However, it has important meaning: Assume $\mathbf{r}(s)$ be a parametrization by arclength parameter. Then by the chain rule and property of arclength parameter, we have

$$\begin{aligned}\mathbf{r}'(t) &= \mathbf{r}'(s)\frac{ds}{dt}\\ &= \mathbf{r}'(s)\|\mathbf{r}'(t)\|.\end{aligned}$$

Since $\|\mathbf{r}'(t)\| \neq 0$, we have

$$\mathbf{r}'(s) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \left( \text{i.e., } \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} = \mathbf{T} \right).$$

Thus $\mathbf{r}(s)$ has always unit speed (i.e., $\mathbf{r}'(s)$ always has a unit length). The two parametrization $(a\cos t, a\sin t)$ and $(a\cos 2\pi t, a\sin 2\pi t)$ have different speeds along the same circle. For the first one, $\mathbf{r}'(t) = (-a\sin t, a\cos t)$. So

$$s(t) = \int_0^t \sqrt{a^2}d\tau = at.$$

So

$$(a\cos t, a\sin t) = (a\cos\frac{s}{a}, a\sin\frac{s}{a}).$$

While for the second one, $\mathbf{r}'(t) = (-2a\pi\sin t, 2a\pi\cos t)$. So

$$s(t) = \int_0^t 2a\pi d\tau = 2a\pi t.$$

Solving $t = s/2a\pi$. So

$$(a\cos 2\pi t, a\sin 2\pi t) = (a\cos\frac{s}{a}, a\sin\frac{s}{a}).$$

So the parametrization by the arc length parameter is the same. In fact, it is independent of any parametrization(Why?)

## 13.4 Curvature and Normal vectors of a Curve

To measure how the curve bends we need to define the following:

**Definition 13.4.1.** The **curvature** of a path $\mathbf{r}$ is the rate of change of unit tangent vector $\mathbf{T}$ per unit of length along the path. In other words,

$$\kappa(t) = \left\|\frac{d\mathbf{T}}{ds}\right\| = \frac{\|d\mathbf{T}/dt\|}{ds/dt} = \frac{1}{\|\mathbf{v}\|}\left\|\frac{d\mathbf{T}}{dt}\right\|.$$

If $\|\frac{d\mathbf{T}}{ds}\|$ is large at some point $P$, the curve turns sharply, and the curvature is large there.

**Example 13.4.2.** Consider a line $\mathbf{r}(t) = \mathbf{a} + t\mathbf{v}$ for some constant vector $\mathbf{a}$. $\mathbf{r}'(t) = \mathbf{v}$, and $\mathbf{T} = \mathbf{v}/\|\mathbf{v}\|$ is a constant vector. So

$$\kappa = 0.$$

## Circular Orbits

Consider a particle of mass $m$ moving at constant speed $s$ in a circular path of radius $r_0$. We can represent its motion (in the plane) as

$$\mathbf{r}(t) = (r_0 \cos t, r_0 \sin t).$$

Since speed is $\|\mathbf{r}'(t)\| = v = r_0$. So the motion is described as

$$\mathbf{v} = \mathbf{r}'(t) = (-r_0 \sin t, r_0 \cos t), \ \|\mathbf{v}\| = r_0.$$

$$\begin{aligned} \mathbf{T} &= \frac{\mathbf{v}}{\|\mathbf{v}\|} = (-\sin t, \cos t) \\ \frac{d\mathbf{T}}{dt} &= (-\cos t, -\sin t) \\ \|\frac{d\mathbf{T}}{dt}\| &= 1. \end{aligned}$$

Hence

$$\kappa = \frac{1}{\|\mathbf{v}\|}\left\|\frac{d\mathbf{T}}{dt}\right\| = \frac{1}{r_0} = \frac{1}{radius}.$$

Since $\mathbf{T}(t)$ is a vector whose length is constant, we have $1 = \|\mathbf{T}(t)\|^2 = \mathbf{T}(t) \cdot \mathbf{T}(t)$. Taking the derivative of constant is zero. Hence

$$0 = \frac{d}{dt}[\mathbf{T}(t) \cdot \mathbf{T}(t)] = \mathbf{T}'(t) \cdot \mathbf{T}(t) + \mathbf{T}(t) \cdot \mathbf{T}'(t) = 2\mathbf{T}(t) \cdot \mathbf{T}'(t).$$

Thus $\mathbf{T}'(t)$ is perpendicular to $\mathbf{T}(t)$ for all $t$.

The vector $d\mathbf{T}/ds$ turns in the direction along which the curve turns.

**Definition 13.4.3.** At a point where $\kappa \neq 0$, the **principal unit normal**

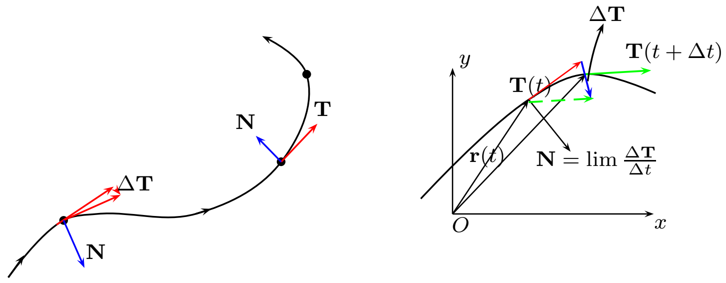



Figure 13.10: **T** turns in the direction of **N**

vector for a smooth curve in the plane is

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|}.$$

The second equality is verified as follows.

$$\begin{aligned}\mathbf{N} &= \frac{d\mathbf{T}/ds}{\|d\mathbf{T}/ds\|} \text{ (use Chain rule)}\\ &= \frac{(d\mathbf{T}/dt)(dt/ds)}{\|d\mathbf{T}/dt\|(dt/ds)}\\ &= \frac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|}.\end{aligned}$$

The vector $\frac{d\mathbf{T}}{ds}$ point in the direction in which **T** turns as the curve bends.

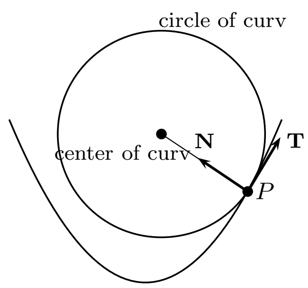



Figure 13.11: Circle of Curvature

## Circle of Curvature for Plane curves

The **circle of curvature** or **osculating circle** at a point $P$ is defined when $\kappa \neq 0$. It is a circle that

(1) has the same tangent line as the curve has

(2) has the same curvature as the curve has

(3) has center in the concave side

The **radius of curvature** of the curve at $P$ is the radius of the circle of curvature. (i.e, $1/\kappa$)

**Example 13.4.4.** Find the osculating circle of parabola $y = x^2$ at the origin.

**sol.** We parameterize the parabola by

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}.$$

Find the osculating circle of parabola $y = x^2$ at the origin.

$$\begin{aligned}
\mathbf{v} &= \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \\
|\mathbf{v}| &= \sqrt{1+4t^2} \\
\mathbf{T} &= \frac{\mathbf{v}}{|\mathbf{v}|} = (1+4t^2)^{-1/2}\mathbf{i} + 2t(1+4t^2)^{-1/2}\mathbf{j}.
\end{aligned}$$

Hence

$$\frac{d\mathbf{T}}{dt} = -4t(1+4t^2)^{-3/2}\mathbf{i} + [2(1+4t^2)^{-1/2} - 8t^2(1+4t^2)^{-3/2}]\mathbf{j}.$$

When $t = 0$,

$$\begin{aligned}
\kappa &= \frac{1}{|\mathbf{v}(0)|}\left|\frac{d\mathbf{T}}{dt}(0)\right| \\
&= \sqrt{0^2 + 2^2} = 2
\end{aligned}$$

Now the normal $\mathbf{N} = \mathbf{j}$. Hence the center is at $(0, 1/2)$ and the circle is

$$(x-0)^2 + (y - \frac{1}{2})^2 = (\frac{1}{2})^2.$$

□

### Curvature and normal vectors for Space curves

The **curvature** and the **principal unit normal** vector for a smooth curve of a space curve given by **r** defined to be the same as plane curve.

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\| = \frac{1}{\|\mathbf{v}\|} \left\| \frac{d\mathbf{T}}{dt} \right\| \tag{13.5}$$

$$\mathbf{N} = \frac{1}{\kappa} \frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|}. \tag{13.6}$$

**Example 13.4.5.** Find the curvature for the helix

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + bt\mathbf{k}, a, b > 0.$$

**sol.**

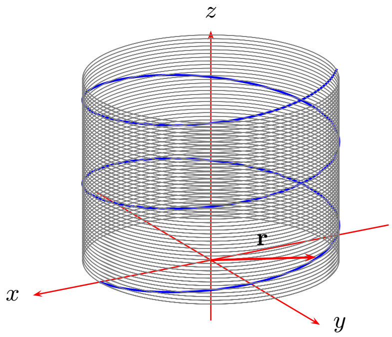
$$\begin{aligned}
\mathbf{v} &= -(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k} \\
|\mathbf{v}| &= \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2+b^2} \\
\mathbf{T} &= \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{a^2+b^2}}[-(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k}].
\end{aligned}$$

Hence

$$\begin{aligned}
\kappa &= \frac{1}{|\mathbf{v}|} \left| \frac{d\mathbf{T}}{dt} \right| \\
&= \frac{1}{\sqrt{a^2+b^2}} \left| \frac{1}{\sqrt{a^2+b^2}}[-(a\cos t)\mathbf{i} - (a\sin t)\mathbf{j}] \right| \\
&= \frac{a}{a^2+b^2} |[-\cos t\mathbf{i} - \sin t\mathbf{j}]| \\
&= \frac{a}{a^2+b^2}.
\end{aligned}$$

□

**Example 13.4.6.** Find the normal **N** for the helix above.




Helix : $(\cos t, \sin t, t)$.

Figure 13.12: Helix

$$
\begin{aligned}
\frac{d\mathbf{T}}{dt} &= -\frac{1}{\sqrt{a^2+b^2}}[(a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}] \\
|\frac{d\mathbf{T}}{dt}| &= \frac{1}{\sqrt{a^2+b^2}}\sqrt{a^2\cos^2 t + b^2\sin^2 t} = \frac{a}{\sqrt{a^2+b^2}} \\
\mathbf{N} &= \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} = -\frac{\sqrt{a^2+b^2}}{a}\frac{1}{\sqrt{a^2+b^2}}[(a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}] \\
&= -(\cos t)\mathbf{i} - (\sin t)\mathbf{j}.
\end{aligned}
$$

Hence **N** is always lying in the $xy$ - plane and pointing toward $z$ axis.

## 13.5 Tangent and Normal components of a

Given a curve, we have seen the unit tangent vector **T** and the unit normal vector **N**. Using these we can define a third vector **B** (called **binormal**, normal to the plane of **T** and **N**) by

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}.$$

The three vectors **T**, **N** and **B** form an orthogonal coordinate system (called **TNB frame** or Frenet (1816-1900) frame) and is useful in studying an object moving on the curve.

We see

$$\begin{aligned}
\mathbf{v} &= \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{T}\frac{ds}{dt} \\
\mathbf{a} &= \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\mathbf{T}\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{dt} \\
&= \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\left(\frac{d\mathbf{T}}{ds}\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\left(\kappa\mathbf{N}\frac{ds}{dt}\right) \\
&= \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}.
\end{aligned}$$

**Definition 13.5.1.** If acceleration vector is written as

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N} \tag{13.7}$$

then

$$a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}|v| \text{ and } a_N = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa|\mathbf{v}|^2 \tag{13.8}$$

are **tangential** and **normal** components of acceleration.

$$\boxed{a_N = \sqrt{\|\mathbf{a}\|^2 - a_T^2}} \tag{13.9}$$

**Example 13.5.2.** Without finding **T** and **N**, write the acceleration of the motion

$$\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j},\ \ t > 0$$

in the form $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$.

**sol.**

$$\begin{aligned}
\mathbf{v} &= \frac{\mathbf{r}}{dt} = (-\sin t + \sin t + t\cos t)\mathbf{i} + (\cos t - \cos t + t\sin t)\mathbf{j} \\
&= (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} \\
|\mathbf{v}| &= \sqrt{t^2\cos^2 t + t^2\sin^2 t} = t \\
a_T &= \frac{d}{dt}\|\mathbf{v}\| = \frac{d}{dt}(t) = 1.
\end{aligned}$$

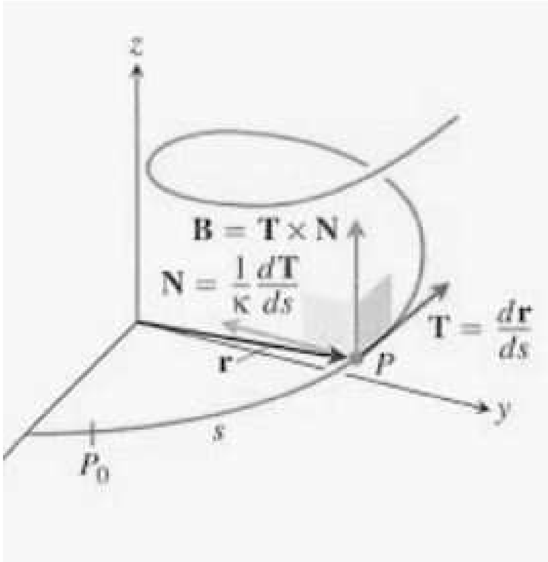



Figure 13.13: Binormal

$$\begin{aligned}
\mathbf{a} &= (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} \\
|\mathbf{a}|^2 &= t^2 + 1 \\
a_N &= \sqrt{\|\mathbf{a}\|^2 - a_T^2} \\
&= \sqrt{t^2 + 1 - 1} = t.
\end{aligned}$$

Thus

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N} = \mathbf{T} + t\mathbf{N}.$$

□

## Torsion

How does $d\mathbf{B}/ds$ behaves in relation to $\mathbf{T}, \mathbf{N}, \mathbf{B}$?

$$\frac{d\mathbf{B}}{ds} = \frac{d(\mathbf{T} \times \mathbf{N})}{ds} = \frac{d\mathbf{T}}{ds} \times \mathbf{N} + \mathbf{T} \times \frac{d\mathbf{N}}{ds} = 0 + \mathbf{T} \times \frac{d\mathbf{N}}{ds}.$$

Thus $d\mathbf{B}/ds$ is orthogonal to $\mathbf{T}$. Since $d\mathbf{B}/ds$ is orthogonal to $\mathbf{B}(\because \mathbf{B} \cdot \mathbf{B} = 1, 0 = d(\mathbf{B} \cdot \mathbf{B})/(ds) = 2\mathbf{B} \cdot d\mathbf{B}/ds)$, it is a scalar multiple of $\mathbf{N}$. Hence we have

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$$

for some scalar $\tau$. This $\tau$ is called **torsion** and one can easily see that it satisfies

$$\boxed{\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}.}$$

(1) $\kappa = |d\mathbf{T}/ds|$ is the rate at which the normal plane turns about the point $P$ as the point moves along the curve.

(2) $\tau = -(d\mathbf{B})/ds)\mathbf{N}$ is the rate at which the osculating plane turns about $\mathbf{T}$ as the point moves along the curve.

## Formula for computing the curvature and torsion

$$\begin{aligned}
\mathbf{v} \times \mathbf{a} &= \left(\frac{ds}{dt}\mathbf{T}\right) \times \left[\frac{d^2 s}{dt^2}\mathbf{T} + \kappa \left(\frac{ds}{dt}\right)^2 \mathbf{N}\right] \\
&= \left(\frac{ds}{dt}\frac{d^2 s}{dt^2}\right)(\mathbf{T} \times \mathbf{T}) + \kappa \left(\frac{ds}{dt}\right)^3 (\mathbf{T} \times \mathbf{N}) \\
&= \kappa \left(\frac{ds}{dt}\right)^3 \mathbf{B}.
\end{aligned}$$

Hence

$$|\mathbf{v} \times \mathbf{a}| = \kappa \left|\frac{ds}{dt}\right|^3 |\mathbf{B}| = \kappa |\mathbf{v}|^3.$$

$$\boxed{\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}} \tag{13.10}$$

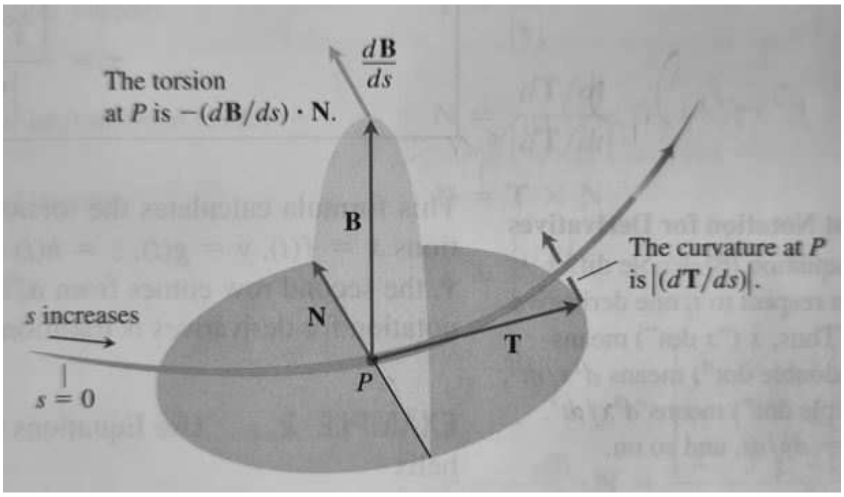



Figure 13.14: TNB